Numerical approach for Hamilton-Jacobi equations on a network: application to traffic

Guillaume Costeseque

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Guillaume Costeseque
(PhD with supervisors R. Monneau & J-P. Lebacque)

Université Paris Est, Ecole des Ponts ParisTech & IFSTTAR

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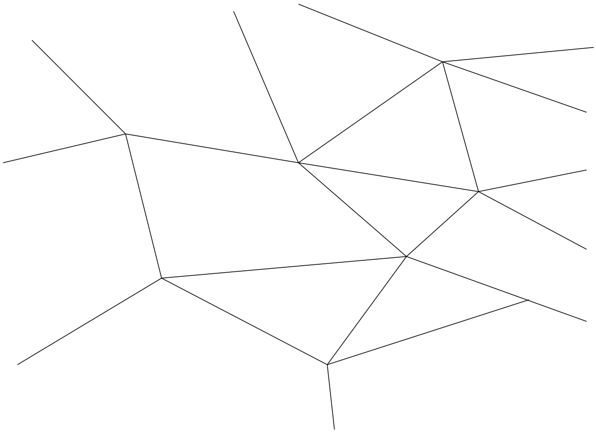
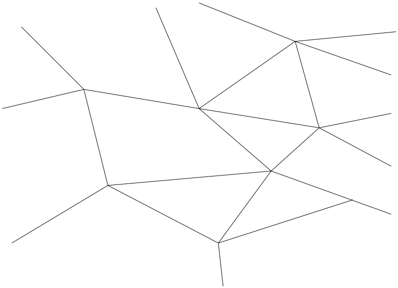
Flows on a network



Flows on a network



A network is like a (oriented) graph made of edges and vertices

Examples:

- traffic flow,
- gas pipelines,
- blood vessels,
- shallow water,
- internet communications...

Outline

Motivation

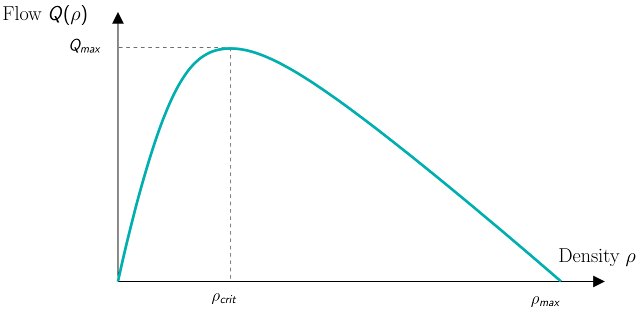
Classical approaches (see A. Bressan's lectures):

- Macroscopic modeling on (homogeneous) sections
- Coupling conditions at (pointwise) junction

For instance, consider

$$\begin{cases} \rho_t + (Q(\rho))_x = 0, & \text{scalar conservation law,} \\ \rho(., t=0) = \rho_0(.), & \text{initial conditions,} \\ \psi(\rho(x=0^-, t), \rho(x=0^+, t)) = 0, & \text{coupling condition.} \end{cases} \tag{1}$$

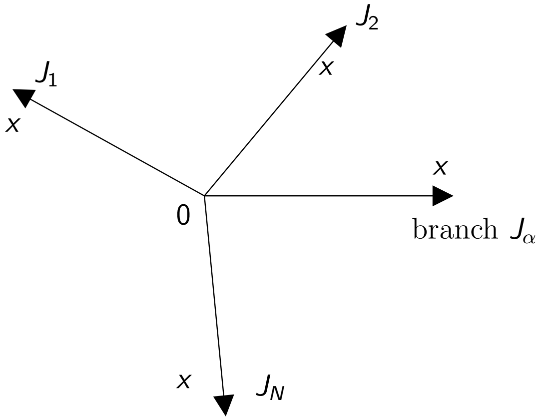
See Garavello, Piccoli [3], Lebacque, Khoshyaran [6] and Bressan et al. [1]



$$Q(\rho) = \rho V(\rho) \quad \text{with} \quad V(\rho) = \text{velocity function}$$

DFs

Star-shaped junction



Junction model

Proposition (Junction model [IMZ, '13])

That leads to the following junction model (see [5])

$$\begin{cases} u_t^\alpha + H_\alpha(u_x^\alpha) = 0, & x > 0,\ \alpha = 1, \dots, N \\ u^\alpha = u^\beta =: u, & x = 0, \\ u_t + \mathcal{H}(u_x^1, \dots, u_x^N) = 0, & x = 0 \end{cases} \tag{2}$$

with initial condition $u^\alpha(0, x) = u_0^\alpha(x)$ *and*

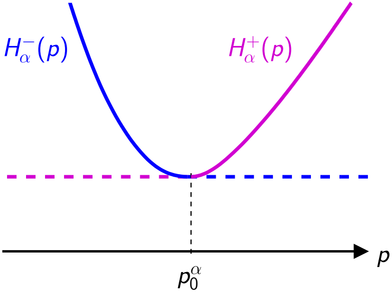
$$\mathcal{H}(u_x^1, \dots, u_x^N) = \underbrace{\max_{\alpha=1,\dots,N} \left\{ H_\alpha^-(u_x^\alpha) \right\}}_{\text{from optimal control}}.$$

Basic assumptions

For all $\alpha = 1, \dots, N$,

(A0) The initial condition u_0^α is Lipschitz continuous.

(A1) The Hamiltonians H_α are $C^1(\mathbb{R})$ and convex such that:



Numerics on networks

Godunov scheme mainly used for conservation laws:

- [Bretti, Natalini, Piccoli '06, '07]: Godunov scheme compared to kinetic schemes / fast algorithms
- [Blandin, Bretti, Cutolo, Piccoli '09]: Godunov scheme adapted for Colombo model (only tested for 1×1 junctions)

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For Hamilton-Jacobi equations on networks:

- [Göttlich, Ziegler, Herty '13]: Lax-Freidrichs scheme outside the junction + coupling conditions (density) at the junction
- [Han, Piccoli, Friesz, Yao '12]: Lax-Hopf formula for HJ equation coupled with a Riemann solver at junction
- [Camilli, Festa, Schieborn '13]: semi-Lagrangian scheme only designed for Eikonal equations

Outline

Presentation of the scheme

Proposition (Numerical Scheme)

Let us consider the discrete space and time derivatives:

$$p_i^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} \quad \text{and} \quad (D_t U)_i^{\alpha,n} := \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}$$

Then we have the following numerical scheme:

$$\begin{cases} (D_t U)_i^{\alpha,n} + \max\{H_\alpha^+(p_{i-1}^{\alpha,n}), H_\alpha^-(p_i^{\alpha,n})\} = 0, & i \geq 1 \\ U_0^n := U_0^{\alpha,n}, & i = 0, \quad \alpha = 1, \ldots, N \\ (D_t U)_0^n + \max_{\alpha=1,\ldots,N} H_\alpha^-(p_0^{\alpha,n}) = 0, & i = 0 \end{cases} \tag{3}$$

With the initial condition $U_i^{\alpha,0} := u_0^\alpha(i\Delta x)$.

Δx and Δt = space and time steps satisfying a CFL condition

CFL condition

The natural CFL condition is given by:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\ldots,N \\ i\geq 0,\ 0\leq n\leq n_T}} \left|H'_\alpha(p_i^{\alpha,n})\right| \tag{4}$$

Gradient estimates

Theorem (Time and Space Gradient estimates)

Assume (A0)-(A1). If the CFL condition (4) is satisfied, then we have that:

(i) *Considering* $M^n = \sup_{\alpha,i} (D_t U)_i^{\alpha,n}$ *and* $m^n = \inf_{\alpha,i} (D_t U)_i^{\alpha,n}$*, we have the following time derivative estimate:*

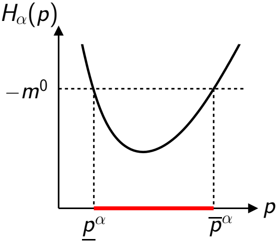
$$m^0 \leq m^n \leq m^{n+1} \leq M^{n+1} \leq M^n \leq M^0$$

(ii) *Considering* $\underline{p}_\alpha = (H_\alpha^-)^{-1}(-m^0)$ *and* $\overline{p}_\alpha = (H_\alpha^+)^{-1}(-m^0)$*, we have the following gradient estimate:*

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \overline{p}_\alpha, \quad \text{for all} \quad i \geq 0, \quad n \geq 0 \quad \text{and} \quad \alpha = 1, ..., N$$

▸ Proof

Stronger CFL condition



As for any $\alpha = 1, \dots, N$, we have that:

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \overline{p}_\alpha \qquad \text{for all} \quad i, n \geq 0$$

Then the CFL condition becomes:

$$\frac{\Delta x}{\Delta t} \geq \sup_{\substack{\alpha=1,\dots,N \\ p_\alpha \in [\underline{p}_\alpha, \overline{p}_\alpha]}} |H'_\alpha(p_\alpha)| \tag{5}$$

Existence and uniqueness

(A2) Technical assumption (Legendre-Fenchel transform)

$$H_\alpha(p) = \sup_{q \in \mathbb{R}} (pq - L_\alpha(q)) \quad \text{with} \quad L''_\alpha \geq \delta > 0, \quad \text{for all index } \alpha$$

Existence and uniqueness

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Theorem (Existence and uniqueness [IMZ, '13])

Under (A0)-(A1)-(A2), there exists a unique viscosity solution u of (2) on the junction, satisfying for some constant $C_T > 0$

$$|u(t,y) - u_0(y)| \leq C_T \quad \textit{for all} \quad (t,y) \in J_T.$$

Moreover the function u is Lipschitz continuous with respect to (t,y).

Convergence

Theorem (Convergence from discrete to continuous [CML, '13])

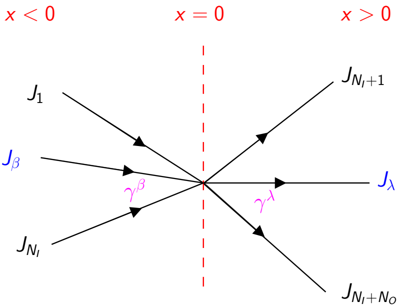
Assume that (A0)-(A1)-(A2) and the CFL condition (5) are satisfied. Then the numerical solution converges uniformly to u the unique viscosity solution of (2) when $\varepsilon \to 0$*, locally uniformly on any compact set* $\mathcal{K}$*:*

$$\limsup_{\varepsilon \to 0} \sup_{(n\Delta t, i\Delta x) \in \mathcal{K}} \left| u^\alpha(n\Delta t, i\Delta x) - U_i^{\alpha,n} \right| = 0$$

▸ Proof

Outline

Setting



N_I incoming and N_O outgoing roads

Car densities

The car density ρ^α solves the LWR equation on branch α:

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0$$

By definition

$$\rho^\alpha = \gamma^\alpha \partial_x U^\alpha \quad \text{on branch } \alpha$$

And

$$\begin{cases} u^\alpha(x,t) = -U^\alpha(-x,t), & x > 0, \text{ for incoming roads} \\ u^\alpha(x,t) = -U^\alpha(x,t), & x > 0, \text{ for outgoing roads} \end{cases}$$

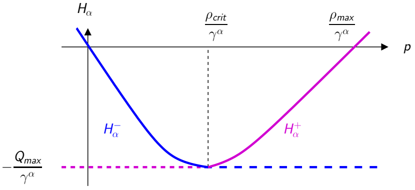
where the car index u^α solves the HJ equation on branch α:

$$u_t^\alpha + H^\alpha(u_x^\alpha) = 0, \quad \text{for } x > 0$$

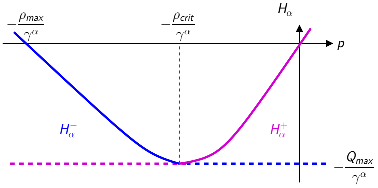
Setting

Flow

$$H_\alpha(p) := \begin{cases} -\dfrac{1}{\gamma^\alpha} Q^\alpha(\gamma^\alpha p) & \text{for} \quad \alpha = 1, \dots, N_I \\ -\dfrac{1}{\gamma^\alpha} Q^\alpha(-\gamma^\alpha p) & \text{for} \quad \alpha = N_I + 1, \dots, N_I + N_O \end{cases}$$



Incoming roads



Outgoing roads

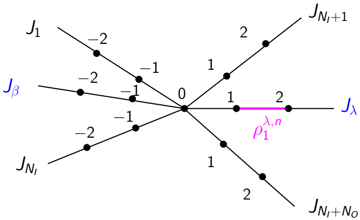
Links with "classical" approach

Definition (Discrete car density)

The discrete car density $\rho_i^{\alpha,n}$ *with* $n \geq 0$ *and* $i \in \mathbb{Z}$ *is given by:*

$$\rho_i^{\alpha,n} := \begin{cases} \gamma^\alpha p_{|i|-1}^{\alpha,n} & \text{for} \quad \alpha = 1, \ldots, N_I, \quad i \leq -1 \\ \\ -\gamma^\alpha p_i^{\alpha,n} & \text{for} \quad \alpha = N_I + 1, \ldots, N_I + N_O, \quad i \geq 0 \end{cases} \tag{6}$$

$x < 0$ $\qquad$ $x > 0$



Traffic interpretation

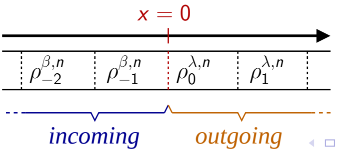
Proposition (Scheme for vehicles densities)

The scheme deduced from (3) for the discrete densities is given by:

$$\frac{\Delta x}{\Delta t}\{\rho_i^{\alpha,n+1} - \rho_i^{\alpha,n}\} = \begin{cases} F^\alpha(\rho_{i-1}^{\alpha,n}, \rho_i^{\alpha,n}) - F^\alpha(\rho_i^{\alpha,n}, \rho_{i+1}^{\alpha,n}) & \text{for } i \neq 0, -1 \\ F_0^\alpha(\rho_0^{\cdot,n}) - F^\alpha(\rho_i^{\alpha,n}, \rho_{i+1}^{\alpha,n}) & \text{for } i = 0 \\ F^\alpha(\rho_{i-1}^{\alpha,n}, \rho_i^{\alpha,n}) - F_0^\alpha(\rho_0^{\cdot,n}) & \text{for } i = -1 \end{cases}$$

With

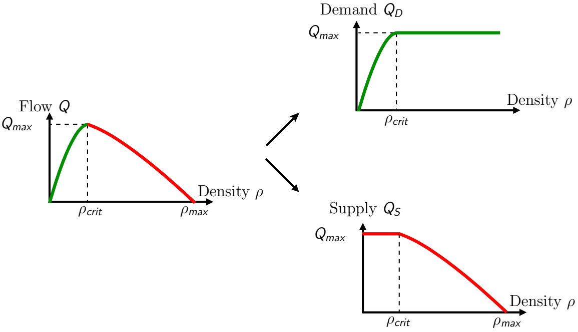
$$\begin{cases} F^\alpha(\rho_{i-1}^{\alpha,n}, \rho_i^{\alpha,n}) := \min\left\{ Q_D^\alpha(\rho_{i-1}^{\alpha,n}),\ Q_S^\alpha(\rho_i^{\alpha,n}) \right\} \\ F_0^\alpha(\rho_0^{\cdot,n}) := \gamma^\alpha \min\left\{ \min\limits_{\beta \le N_I} \dfrac{1}{\gamma^\beta} Q_D^\beta(\rho_0^{\beta,n}),\ \min\limits_{\lambda > N_I} \dfrac{1}{\gamma^\lambda} Q_S^\lambda(\rho_0^{\lambda,n}) \right\} \end{cases}$$



Supply and demand functions

Remark

It recovers the seminal Godunov scheme with passing flow = minimum between upstream demand Q_D and downstream supply Q_S.



From [Lebacque '93, '96]

Supply and demand VS Hamiltonian

$$H_\alpha^-(p) = \begin{cases} -\dfrac{1}{\gamma^\alpha} Q_D^\alpha(\gamma^\alpha p) & \text{for} \quad \alpha = 1, \ldots, N_I \\ -\dfrac{1}{\gamma^\alpha} Q_S^\alpha(-\gamma^\alpha p) & \text{for} \quad \alpha = N_I + 1, \ldots, N_I + N_O \end{cases}$$

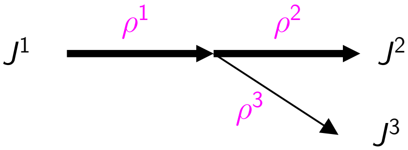
And

$$H_\alpha^+(p) = \begin{cases} -\dfrac{1}{\gamma^\alpha} Q_S^\alpha(\gamma^\alpha p) & \text{for} \quad \alpha = 1, \ldots, N_I \\ -\dfrac{1}{\gamma^\alpha} Q_D^\alpha(-\gamma^\alpha p) & \text{for} \quad \alpha = N_I + 1, \ldots, N_I + N_O \end{cases}$$

Outline

Example of a Diverge

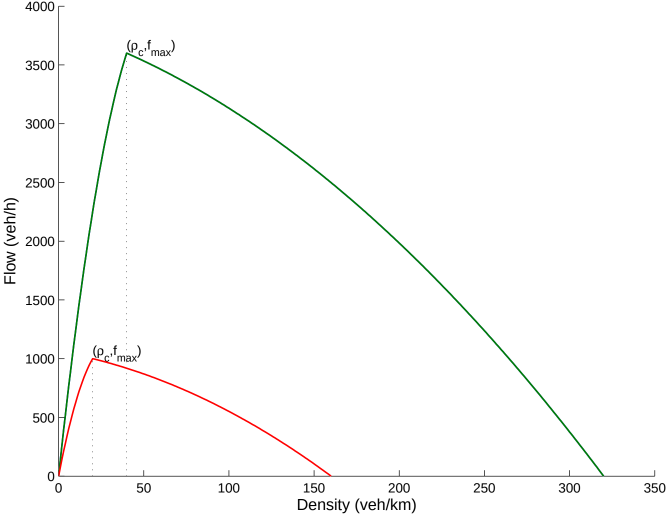
An off-ramp:



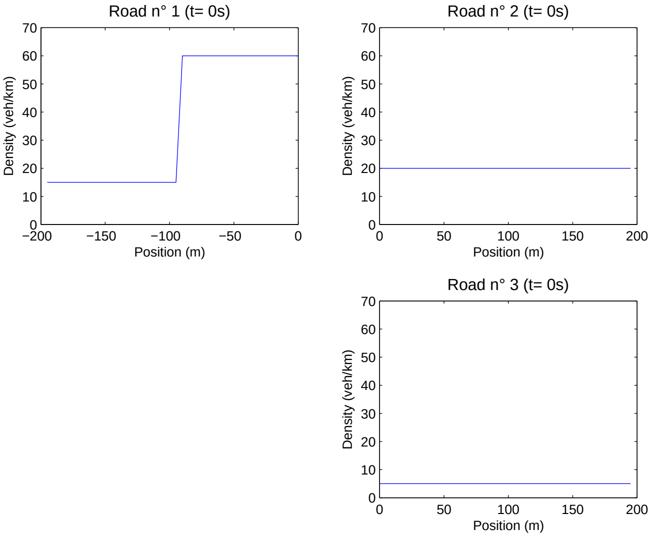
with

$$\begin{cases} \gamma^e = 1, \\ \gamma^l = 0.75, \\ \gamma^r = 0.25 \end{cases}$$

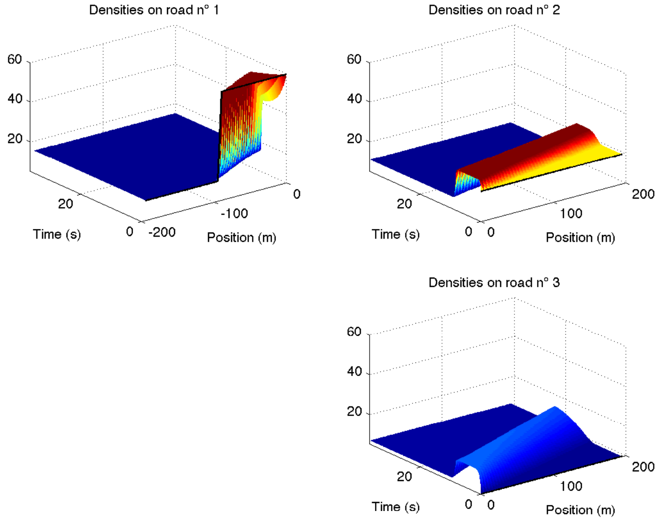
Fundamental Diagrams



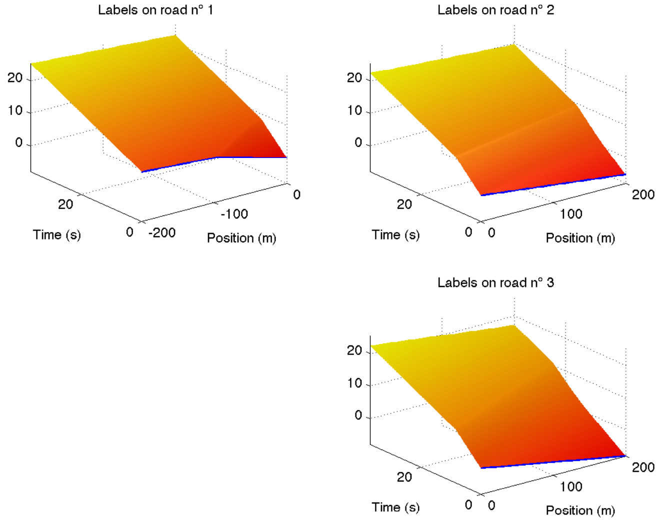
Initial conditions (t=0s)



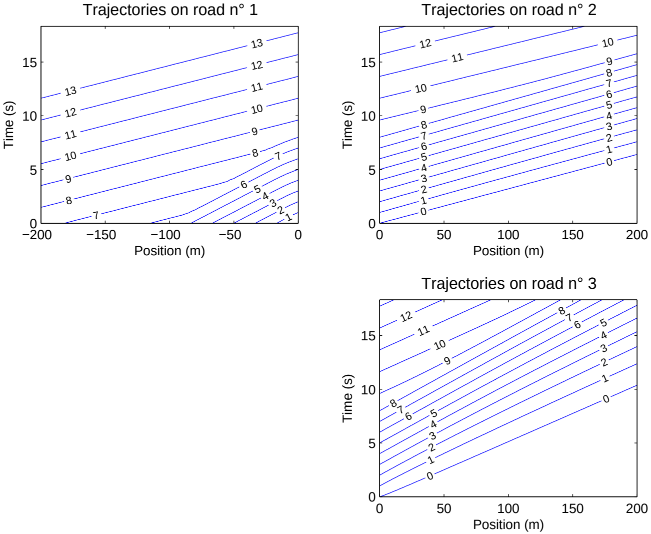
Numerical solution: densities



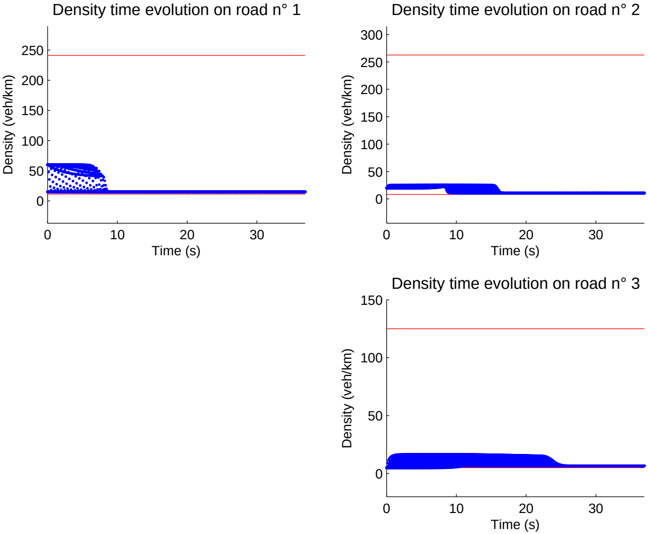
Numerical solution: Hamilton-Jacobi



Trajectories



Gradient estimates



Outline

New junction model

Proposition (Junction model [IM, '14])

From [4], we have

$$\begin{cases} u_t^\alpha + H_\alpha(u_x^\alpha) = 0, & x > 0, \ \alpha = 1, \ldots, N \\ u^\alpha = u^\beta =: u, & x = 0, \\ u_t + \mathcal{H}(u_x^1, \ldots, u_x^N) = 0, & x = 0 \end{cases} \tag{7}$$

with initial condition $u^\alpha(0,x) = u_0^\alpha(x)$ *and*

$$\mathcal{H}(u_x^1, \ldots, u_x^N) = \max\Big[\overbrace{\mathcal{L}}^{\text{flux limiter}}, \underbrace{\max_{\alpha=1,\ldots,N} \left\{ H_\alpha^-(u_x^\alpha) \right\}}_{\substack{\text{minimum between} \\ \text{demand and supply}}} \Big].$$

Weaker assumptions on the Hamiltonians

For all $\alpha = 1, \dots, N$,
(A0) The initial condition u_0^α is Lipschitz continuous.
(A1) The Hamiltonians H_α are continuous and quasi-convex i.e. there exists points p_0^α such that

$$\begin{cases} H_\alpha & \text{is non-increasing on} \quad (-\infty, p_0^\alpha], \\ \\ H_\alpha & \text{is non-decreasing on} \quad [p_0^\alpha, +\infty). \end{cases}$$

Homogenization on a network

Proposition (Homogenization on a periodic network [IM'14])

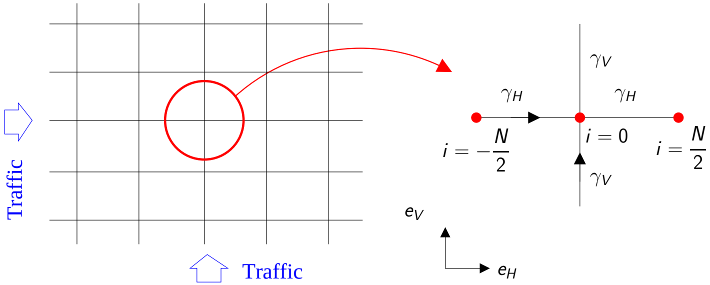
Assume (A0)-(A1). Consider a periodic network.
If u^ε satisfies (oscillating) HJ equation on network,
then u^ε converges uniformly towards u^0 when $\varepsilon \to 0$,
with u^0 solution of

$$u_t^0 + \overline{H}\left(\nabla_x u^0\right) = 0, \quad t > 0, \ x \in \mathbb{R}^d \tag{8}$$

See Prof. R. Monneau's lecture and [4]

Numerical homogenization on a network

Numerical scheme adapted to the cell problem



First example

Proposition (Effective Hamiltonian for fixed coefficients [IM'14])

If (γ^H, γ^V) are fixed, then the

- *(Hamiltonian) effective Hamiltonian $\overline{H}$ is given by*

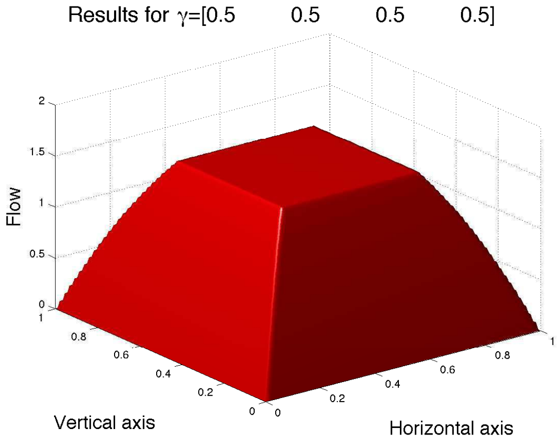
$$\overline{H}(u_{H,x}, u_{V,x}) = \max \left\{ \mathcal{L}, \max_{i=\{H,V\}} H(u_{i,x}) \right\},$$

- *(traffic flow) effective flow $\overline{Q}$ is given by*

$$\overline{Q}(\rho_H, \rho_V) = \min \left\{ -\mathcal{L}, \frac{Q(\rho_H)}{\gamma^H}, \frac{Q(\rho_V)}{\gamma^V} \right\}.$$

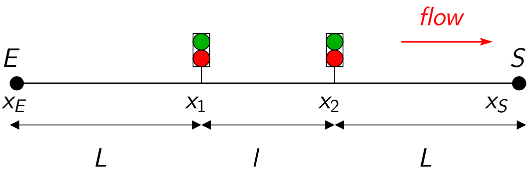
First example

Numerics: assume $Q(\rho) = 4\rho(1-\rho)$ and $\mathcal{L} = -1.5$,



Second example

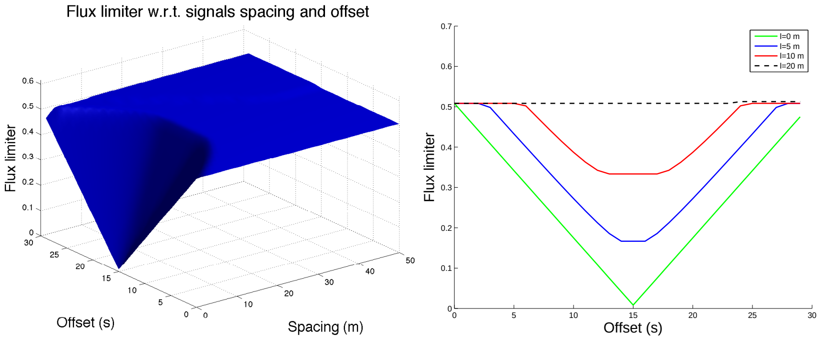
Two consecutive traffic signals on a 1D road



Homogenization theory by [G. Galise, C. Imbert, R. Monneau, '14]

Second example

Effective flux limiter $-\overline{\mathcal{L}}$ (numerics only)



Thanks for your attention

guillaume.costeseque@cermics.enpc.fr
guillaume.costeseque@ifsttar.fr

Some references I

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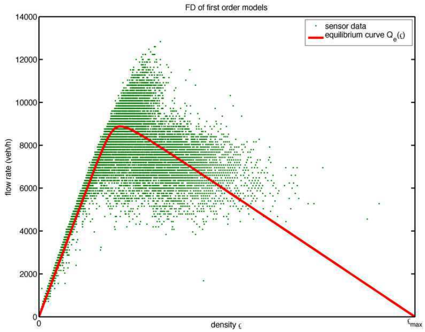
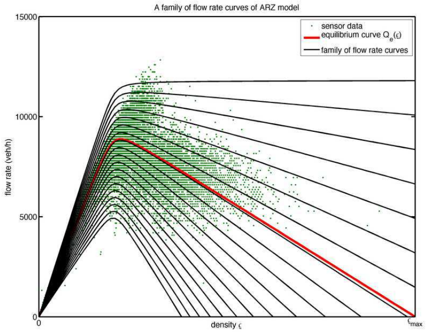
J.-P. Lebacque and M. M. Khoshyaran, *First-order macroscopic traffic flow models: Intersection modeling, network modeling*, in Transportation and Traffic Theory. Flow, Dynamics and Human Interaction. 16th International Symposium on Transportation and Traffic Theory, 2005.

Fundamental diagram

Fundamental diagram: multi-valued in congested case

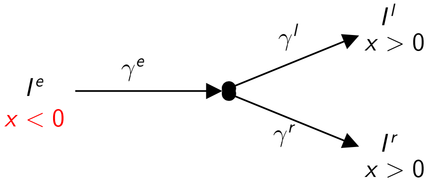




[S. Fan, M. Herty, B. Seibold, 2013], NGSIM dataset

Back

Motivation: the simple divergent road



$$\begin{cases} \gamma^e = 1, \\ 0 \leq \gamma^l, \gamma^r \leq 1, \\ \gamma^l + \gamma^r = 1 \end{cases}$$

LWR model [Lighthill, Whitham '55; Richards '56] on each branch α:

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0$$

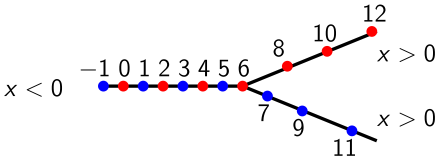
Getting the Hamilton-Jacobi equation

LWR model on each branch (outside the junction point)

$$\rho_t^\alpha + (Q^\alpha(\rho^\alpha))_x = 0 \quad \text{on branch} \quad \alpha$$

Primitive:

$$\begin{cases} U^\alpha(x,t) = U^\alpha(0,t) + \dfrac{1}{\gamma^\alpha}\displaystyle\int_0^x \rho^\alpha(y,t)dy, \\ U^\alpha(0,t) = g(t) = \text{index of the single car at the junction point} \end{cases}$$



Getting the Hamilton-Jacobi equation

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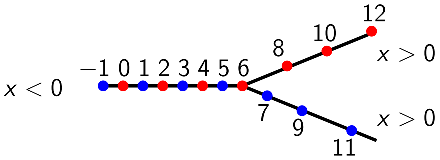

$$U_t^\alpha + \frac{1}{\gamma^\alpha} Q^\alpha(\gamma^\alpha U_x^\alpha) = g'(t) + \frac{1}{\gamma^\alpha} Q^\alpha(\rho^\alpha(0,t))$$

$$= 0 \quad \text{for a good choice of } g$$

Back

Sketch of the proof (gradient estimates):

Time derivative estimate:

1. Estimate on $m^{\alpha,n} = \inf_i \, (D_t U)_i^{\alpha,n}$ and partial result for $m^n = \inf_\alpha \, m^{\alpha,n}$
2. Similar estimate for M^n
3. Conclusion

Space derivative estimate:

1. New bounded Hamiltonian $\tilde{H}_\alpha(p)$ for $p \leq \underline{p}_\alpha$ and $p \geq \overline{p}_\alpha$
2. Time derivative estimate from above
3. Lemma: if for any (i, n, α), $(D_t U)_i^{\alpha,n} \geq m^0$ then

$$\underline{p}_\alpha \leq p_i^{\alpha,n} \leq \overline{p}_\alpha$$

4. Conclusion as $\tilde{H}_\alpha = H_\alpha$ on $[\underline{p}_\alpha, \overline{p}_\alpha]$

See [2]

Back

Convergence with uniqueness assumption

Sketch of the proof: (Comparison principle very helpful)

1. $\overline{u}^{\alpha}(t,x) := \limsup_{\varepsilon} U_i^{\alpha,n}$ is a subsolution of (2) (contradiction on Definition inequality with a test function φ)
2. Similarly, $\underline{u}^{\alpha}$ is a supersolution of (2)
3. Conclusion: $\overline{u}^{\alpha} = \underline{u}^{\alpha}$ viscosity solution of (2)

See [2]

Convergence without uniqueness assumption

Sketch of the proof: (No comparison principle)

1. Discrete Lipschitz bounds on $u_\varepsilon^\alpha(n\Delta t, i\Delta x) := U_i^{\alpha,n}$
2. Extension by continuity of u_ε^α
3. Ascoli theorem (convergent subsequence on every compact set)
4. The limit of one convergent subsequence $(u_\varepsilon^\alpha)_\varepsilon$ is super and sub-solution of (2)

See [2]

Back